

ON THE STRONG COUPLING LIMIT OF THE FADDEEV-HOPF MODEL

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ABSTRACT. The variational calculus for the Faddeev-Hopf model on a general Riemannian domain, with general Kähler target space, is studied in the strong coupling limit. In this limit, the model has key similarities with pure Yang-Mills theory, namely conformal invariance in dimension 4 and an infinite dimensional symmetry group. The first and second variation formulae are calculated and several examples of stable solutions are obtained. In particular, it is proved that all immersive solutions are stable. Topological lower energy bounds are found in dimensions 2 and 4. An explicit description of the spectral behaviour of the Hopf map $S^3 \rightarrow S^2$ is given, and a conjecture of Ward concerning the stability of this map in the full Faddeev-Hopf model is proved.

1. INTRODUCTION

Theoretical physics has long been a rich source of geometrically interesting and natural variational problems. The Yang-Mills equations, of deep significance for the differential topology of 4-manifolds [4], and the Yang-Mills-Higgs equations, which have led to interesting results in hyperkähler geometry [1], both originated in elementary particle physics. Harmonic map theory, while not originating in theoretical physics, has found many applications in high energy and condensed matter physics, with physicists frequently contributing genuinely new insights.

The purpose of this paper is to present a systematic study of a variational problem arising in the so-called Faddeev-Hopf (or Faddeev-Skyrme) model [6], originally proposed as a model of quark confinement (among other phenomena) in high energy physics. Let M be some Riemannian manifold, representing physical space, and N a Kähler manifold, the target space, with Kähler form ω . The model has a single field $\phi : M \rightarrow N$, the energy functional (or action functional, in the case where M is euclideanized spacetime after Wick rotation) being

$$\mathcal{E}(\phi) = \frac{1}{2} \int_M (|d\phi|^2 + \alpha |\phi^* \omega|^2),$$

$\alpha \geq 0$ being a coupling constant. The model of original interest has $M = \mathbb{R}^3$, $N = S^2$. The weak coupling limit of this model, $\alpha = 0$ has of course been intensively studied: it is the harmonic map problem. This is conformally invariant if M has dimension 2. By contrast, we shall study the strong coupling limit, $\alpha \rightarrow \infty$, or more precisely, the variational problem for the energy functional

$$E(\phi) = \lim_{\alpha \rightarrow \infty} \alpha^{-1} \mathcal{E}(\phi) = \frac{1}{2} \int_M |\phi^* \omega|^2.$$

This does not seem to have received systematic study in either the theoretical physics or differential geometry communities. It has been studied in the specific case $M = \mathbb{R} \times S^3$ (with a Lorentzian metric, actually) and $N = S^2, \mathbb{C}$ or the

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hyperbolic plane by de Carli and Ferreira [3]. It has some important similarities with pure Yang-Mills theory. It is invariant under an infinite dimensional group of symmetries, the group of symplectic diffeomorphisms of N , rather as Yang-Mills theory is invariant under gauge transformations. It is also, as we will demonstrate, conformally invariant if M has dimension 4. Both these facts were known to de Carli and Ferreira in the specific context they studied. The most interesting situation physically is when $M = S^4$, interpreted as the conformal compactification of \mathbb{R}^4 . Nontrivial solutions in this case may receive the physical interpretation of instantons in the strong coupling limit of the Faddeev-Hopf model in $(3 + 1)$ dimensions, just as critical points of the Yang-Mills functional on S^4 are interpreted as pure gauge-theory instantons. Such solutions have profound effects on the quantized version of the field theory [13, ch10].

Our motivation for studying this variational problem is twofold. First, simple curiosity prompts us to ask what the geometric character of the variational calculus for this functional is. We will see that both the first and second variation formulae can be given elegant and natural geometric formulations from which strong results quickly follow. For example, we will show that all immersive solutions are stable, and that there are no non-vacuum (i.e., $E > 0$) immersive solutions in the case $M = S^4$, for any choice of target space. Second, we hope that studying one term in the Faddeev-Hopf model in isolation will give valuable insight into the finite coupling model. Indeed, we will identify a large class of critical points of E which are also harmonic maps, and hence critical points of \mathcal{E} for all α . In particular, we are able to prove a stability conjecture of Ward concerning the full Faddeev-Hopf model on $M = S^3$ [17]. The functional E also arises as one term in the so-called baby Skyrme models studied by Zakrzewski and collaborators [12], and our results should find applications in these models too.

The rest of the paper is structured as follows. In section 2 we carefully define the functional E , prove that it is conformally invariant in dimension 4, derive the first variation formula (Euler-Lagrange equation) for ϕ and construct some interesting explicit solutions. In section 3 we consider submersive solutions in particular, identifying a large class of critical submersions which are also harmonic. In section 4 we obtain topological lower bounds on E when M has dimension 2 or 4, and derive the second variation formula in the general case. The results are used to prove the stability of several interesting solutions. Finally, in section 5 the variational calculus for the projection $G \rightarrow G/K$ onto a Hermitian symmetric space is developed in general, and the results used to show that the Hopf map $S^3 \rightarrow S^2$ in particular is stable. A proof of Ward's conjecture quickly follows from this.

2. THE FIRST VARIATION

In this section we assume that (M^m, g) is a compact, oriented Riemannian manifold of dimension m . For any vector bundle E over M , we denote by $\Gamma(E)$ the space of sections of E .

Recall that the metric g on M induces a (pointwise) metric on the bundle of p -forms on M , defined by

$$\langle \alpha, \beta \rangle = \frac{1}{p!} \sum_{i_1, \dots, i_p=1}^m \alpha(e_{i_1}, \dots, e_{i_p}) \beta(e_{i_1}, \dots, e_{i_p}),$$

where e_1, \dots, e_m is a local orthonormal frame on M . By using the Hodge $*$ -operator, we get the relation

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle * 1$$

for any two p -forms α and β ; here $*1$ is the volume element on M . Integrating this inner product over M gives a global L^2 -product

$$\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha, \beta \rangle * 1 = \int_M \alpha \wedge * \beta \quad (\alpha, \beta \in \Gamma(\wedge^p T^* M)),$$

with corresponding norm $\|\alpha\|_{L^2}^2 = \langle \alpha, \alpha \rangle_{L^2}$. With respect to this L^2 -product, the exterior differentiation operator

$$d : \Gamma(\wedge^p T^* M) \rightarrow \Gamma(\wedge^{p+1} T^* M)$$

has the adjoint operator

$$\delta : \Gamma(\wedge^p T^* M) \rightarrow \Gamma(\wedge^{p-1} T^* M), \quad \delta \alpha = (-1)^{m+mp+1} * d * \alpha.$$

We will also be using the *musical isomorphisms* on M which are defined as follows:

$$\flat : \Gamma(TM) \rightarrow \Gamma(T^* M), \quad \flat X = g(X, \cdot), \quad \sharp = \flat^{-1}.$$

Let (N^n, h, J) be a compact Kähler manifold of real dimension n and with Kähler form $\omega = h(J\cdot, \cdot)$. For a smooth map $\phi : M \rightarrow N$ we define the *energy functional*

$$E(\phi) = \frac{1}{2} \|\phi^* \omega\|_{L^2}^2 = \frac{1}{2} \int_M \phi^* \omega \wedge * \phi^* \omega.$$

Any map ϕ for which $E(\phi) = 0$, the minimum possible, will be called a *vacuum solution* or *vacuum* of the theory. Clearly ϕ is a vacuum if and only if $\phi^* \omega = 0$ everywhere, that is, if ϕ is isotropic.

We begin our investigation of $E(\phi)$ by verifying that it is, like the Yang-Mills functional, invariant under conformal changes of g if M has dimension 4.

Proposition 2.1. *Assume that $\phi : M \rightarrow N$ is a map from a 4-dimensional Riemannian manifold to a Kähler manifold. Then the functional $E(\phi)$, and therefore also the Euler-Lagrange equation for ϕ , is invariant under conformal changes of the metric on M .*

Proof. Assume that g is a Riemannian metric on M and that $\tilde{g} = \lambda^2 g$, where λ is some positive function on M . Denote by $\langle \cdot, \cdot \rangle_{\tilde{g}}$ the metric induced by \tilde{g} on 2-forms and let $\tilde{*}1$ be the corresponding volume element. Then

$$\tilde{*}1 = \lambda^4 * 1 \quad \text{and} \quad \langle \alpha, \alpha \rangle_{\tilde{g}} = \lambda^{-4} \langle \alpha, \alpha \rangle,$$

for any 2-form α on M . Hence the form $\langle \alpha, \alpha \rangle * 1$ remains unchanged. \square

Next we derive the Euler-Lagrange equation for $E(\phi)$ in the general case.

Proposition 2.2. *For a smooth variation $\phi_t : M \rightarrow N$ of ϕ with variational vector field $X \in \Gamma(\phi^{-1}TN)$, we have*

$$\frac{d}{dt}E(\phi_t)|_{t=0} = \int_M \omega(X, d\phi(\sharp \delta \phi^* \omega)) * 1.$$

For the proof, let us recall the following simple result.

Lemma 2.3 (Homotopy Lemma). *Let M and N be two manifolds and*

$$\phi_t : M \rightarrow N$$

a smooth family of maps. For any closed 2-form η on N we have

$$\frac{\partial}{\partial t} \phi_t^* \eta = d(\phi_t^* \iota(\frac{\partial \phi_t}{\partial t}) \eta).$$

Here ι denotes the interior product. For a proof of this lemma see, e.g., [5, p49].

Proof of Proposition 2.2. It is obvious that $\frac{\partial}{\partial t}$ commutes with the Hodge $*$ -operator on $\wedge^* T^*M$. By the Homotopy Lemma we have

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \Big|_{t=0} \phi_t^* \omega \wedge * \phi_t^* \omega &= \frac{1}{2} \left(\frac{\partial}{\partial t} \phi_t^* \omega \right) \wedge * \phi_t^* \omega + \frac{1}{2} \phi_t^* \omega \wedge * \left(\frac{\partial}{\partial t} \phi_t^* \omega \right) \\ &= \frac{1}{2} d\phi^* \iota_X \omega \wedge * \phi_t^* \omega + \frac{1}{2} \phi_t^* \omega \wedge * \phi^* \iota_X \omega \\ &= \langle d\phi^* \iota_X \omega, \phi^* \omega \rangle * 1. \end{aligned}$$

Therefore

$$\frac{\partial}{\partial t} E(\phi_t) \Big|_{t=0} = \int_M \langle \phi^* \iota_X \omega, \delta \phi^* \omega \rangle * 1,$$

and

$$\langle \phi^* \iota_X \omega, \delta \phi^* \omega \rangle = \phi^* \iota_X \omega(\sharp \delta \phi^* \omega) = \omega(X, d\phi(\sharp \delta \phi^* \omega)).$$

This proves the proposition. \square

Corollary 2.4. *The map $\phi : M \rightarrow N$ is a critical point for the functional if and only if*

$$\sharp \delta \phi^* \omega \in \ker d\phi$$

everywhere on M .

Example 2.5. Assume that $M = N$ and $\phi : N \rightarrow N$ is the identity map. Then $\phi^* \omega = \omega$, and $\delta \omega = 0$. Hence ϕ is a critical point for the functional.

Remark 2.6. Assume that ϕ is smooth and is immersive on a dense set, that is, for all x in a dense subset of M , the differential

$$d\phi_x : T_x M \rightarrow T_{\phi(x)} N$$

is injective. By the Corollary, if ϕ is a critical point of the functional, then the 1-form $\delta \phi^* \omega$ vanishes almost everywhere, and hence vanishes everywhere by continuity. Hence the 2-form $\phi^* \omega$ is co-closed. Since it is obviously closed, we see that *an almost everywhere immersive map is a critical point of the functional if and only if $\phi^* \omega$ is a harmonic 2-form.*

In particular, when $H^2(M, \mathbb{R}) = 0$, the only immersive critical points defined on M are *isotropic* immersions, i.e., maps for which $\phi^*\omega = 0$. As previously remarked, such a map has $E(\phi) = 0$, and hence is a vacuum solution of the field theory. The set of vacuum solutions of this theory is unusually rich.

Example 2.7. The map $\phi : S^4 \rightarrow \mathbb{C}P^4$, defined as the 2-fold covering by S^4 of $\mathbb{R}P^4$ followed by the natural embedding of $\mathbb{R}P^4$ to $\mathbb{C}P^4$ is clearly isotropic, hence a vacuum. An interesting question is whether this vacuum is path connected, through vacua, to the trivial vacuum $\phi = \text{constant}$. One suspects not.

Certainly the set of vacua may fail to be path connected. Consider the zero section of TS^n , equipped with the Stenzel metric [15]. This is manifestly an isotropic immersion $i : S^n \rightarrow TS^n$, hence a vacuum. Given any smooth map $\phi : S^n \rightarrow S^n$, the composition $i \circ \phi$ is still isotropic, hence a vacuum. But if the degree of ϕ is not unity, then i and $i \circ \phi$ are not even homotopic, much less path connected through isotropic maps. Hence the set of vacua in the case $M = S^n$ and $N = TS^n$ is not path connected.

Of primary physical interest, given their physical interpretation as instantons, are smooth anisotropic critical points on $M = S^4$ which minimize $E(\phi)$ within their homotopy class. In particular, one would like a smooth anisotropic minimizer in the nontrivial class of $\pi_4(S^2)$, a *pure Faddeev-Hopf instanton*. In fact, it remains an open question whether smooth anisotropic critical points exist on S^4 at all, for any choice of target space. A standard starting point for finding special solutions is the use of symmetry reduction. A fundamental difficulty in exploiting symmetry reduction is raised by Remark 2.6: symmetry reductions of the variational problem on S^4 tend to produce only maps which are either trivial or immersive. The best we have managed is a smooth solution mapping the punctured hemisphere into $\mathbb{C}P^2$, two copies of which can be glued together to give a continuous map $S^4 \rightarrow \mathbb{C}P^2$ which, away from the poles and the equator of S^4 , is smooth and satisfies the field equation, and has finite total energy. This map is constructed in the next example.

Example 2.8. Let $M = S^4$ and $N = \mathbb{C}P^2$. The twice punctured 4-sphere is conformally equivalent to the cylinder $\mathbb{R} \times \mathbf{SU}(2)$ given the metric

$$g = dt^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2,$$

where σ_i are the usual left-invariant one forms on $\mathbf{SU}(2)$. So we may seek critical points on $\mathbb{R} \times \mathbf{SU}(2)$ satisfying appropriate boundary conditions as $|t| \rightarrow \infty$. Let $I = (a, b) \subset \mathbb{R}$ be any open interval and Q be the Banach manifold of, for example, C^2 maps $I \times \mathbf{SU}(2) \rightarrow \mathbb{C}P^2$. Then $E : Q \rightarrow \mathbb{R}$ is C^1 , and there is a natural action of $\mathbf{SU}(2) \times V_4$ on Q given by

$$\begin{aligned} \phi(t, X) &\xrightarrow{(U,1)} \begin{bmatrix} U & \\ & 1 \end{bmatrix} \phi(t, UX) \\ \phi(t, X) &\xrightarrow{(1,P_1)} \begin{bmatrix} X & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} X^{-1} & \\ & 1 \end{bmatrix} \phi(t, UX) \\ \phi(t, X) &\xrightarrow{(1,P_2)} \overline{\phi(t, \overline{X})}, \end{aligned}$$

where P_1, P_2 generate the Viergruppe $V_4 = \{1, P_1, P_2, P_1P_2\}$.

The set Q_0 of fixed points of this action consists of maps of the form

$$\phi(t, X) = \begin{bmatrix} X & \\ & 1 \end{bmatrix} [\alpha(t) : 0 : 1],$$

where $\alpha : I \rightarrow S^1 = \mathbb{R} \cup \{\infty\}$. Since $\mathbf{SU}(2) \times V_4$ is compact, we may apply the principle of symmetric criticality [11] to deduce that any critical point of $E|_{Q_0}$ is automatically a critical point of E . Routine calculation shows that

$$E|_{Q_0}(\alpha) = \pi^2 \int_I \frac{2\alpha^2 \dot{\alpha}^2}{(1 + \alpha^2)^4} + \frac{\alpha^4}{2(1 + \alpha^2)^2} dt,$$

where $\dot{}$ denotes differentiation with respect to t . This may be thought of as the action of a one-dimensional Lagrangian mechanical system. By invariance under t translation, all solutions $\alpha(t)$ conserve the quantity

$$H = \frac{\alpha^2 \dot{\alpha}^2}{(1 + \alpha^2)^4} - \frac{\alpha^4}{4(1 + \alpha^2)^2}.$$

If we wish ϕ to extend to S^4 we should insist that $\alpha, \dot{\alpha} \rightarrow 0$ as $t \rightarrow \infty$, so only solutions with $H = 0$ are of interest. The $H = 0$ level curve in the $(\alpha, \dot{\alpha})$ plane is, $4\dot{\alpha}^2 = \alpha^2(1 + \alpha^2)^2$, whence one finds solutions $\alpha_+ : (0, \infty) \rightarrow \mathbb{R}$ and $\alpha_- : (-\infty, 0) \rightarrow \mathbb{R}$,

$$\alpha_+(t) = \frac{1}{\sqrt{e^t - 1}}, \quad \alpha_-(t) = \frac{-1}{\sqrt{e^{-t} - 1}}.$$

Gluing these together gives a continuous map $S^4 \rightarrow \mathbb{CP}^2$,

$$\phi(t, X) = \begin{bmatrix} X & \\ & 1 \end{bmatrix} [1 : 0 : \frac{t}{|t|} \sqrt{e^{|t|} - 1}],$$

of total energy $E = \pi^2$, which away from $t = \pm\infty$ and $0 \times \mathbf{SU}(2)$, is smooth and solves the field equation. Clearly this solution fails to be globally smooth.

Remark 2.9. One can find global, smooth, anisotropic, finite energy critical maps on \mathbb{R}^4 if one equips it with a metric outside the Euclidean conformal class. An example is given in the next section, Example 3.2.

Remark 2.10. De Carli and Ferreira have constructed an ingenious symmetry reduction in the case $M = \mathbb{R} \times S^3$ (Lorentzian in their original version, but the same reduction works in the Riemannian case) and $N = S^2$, by imposing invariance under a $T^2 \times \mathbb{Z}_2$ group of symmetries [3]. This reduces the field equation, not to a nonlinear ODE as in the example above, but to a *linear* elliptic PDE for a single real function on \mathbb{R}^2 . Unfortunately, only solutions which remain bounded on the whole of \mathbb{R}^2 give globally well defined maps ϕ on S^4 , and no such nontrivial solutions exist in the Riemannian case.

3. CRITICAL SUBMERSIONS

In light of Corollary 2.4 and Remark 2.6 it is natural to seek critical maps in the case where the dimension of M exceeds that of N . In particular, there exists a large number of interesting critical submersions. Such solutions automatically have non-vanishing energy, so are not vacua. We begin with the simple example

of projection on a (possibly) warped product, then reformulate the first variation formula in a way better suited to submersions.

Example 3.1. Assume that (P, k) and (N, h) are two compact Riemannian manifolds and that $f : P \rightarrow \mathbb{R}$ is a positive, smooth function. The *warped product* of (P, k) and (N, h) by f is the manifold $P \times N$ with the Riemannian metric

$$g = k + f^2 h.$$

Assume further that (N, h, ω) is Kähler. Then the projection map $\phi : P \times N \rightarrow N$ onto the second coordinate is critical.

To prove this, let $*_P$ and $*_N$ be the Hodge star operators on P and N , respectively, so that the volume form on P is $*_P 1$. Then, as is easily seen,

$$*\phi^* \omega = f^{2(n-2)} *_P 1 \wedge *_N \omega = \frac{f^{2(n-2)}}{(n-1)!} *_P 1 \wedge \omega^{n-1},$$

where $n = \dim_{\mathbb{C}} N$. As f is a function only on P , $df \wedge *_P 1 = 0$. Furthermore, $*_P 1$ and ω^{n-1} are obviously closed. Hence $\delta\phi^* \omega = 0$.

Example 3.2. One can use projection on a product to construct global, smooth, finite energy solutions on $M = \mathbb{R}^4$ if one equips it with a metric outside the Euclidean conformal class. For example, let

$$g = (1 + x_3^2 + x_4^2)^2(dx_1^2 + dx_2^2) + (1 + x_1^2 + x_2^2)^2(dx_3^2 + dx_4^2).$$

Then (\mathbb{R}^4, g) is complete, has infinite volume and is Ricci positive with bounded scalar curvature. It is conformally equivalent to $S_{\times}^2 \times S_{\times}^2$ where S_{\times}^2 is the punctured unit sphere. In terms of a stereographic coordinate on S_{\times}^2 projected from the puncture, the equivalence is $x \equiv (x_1 + ix_2, x_3 + ix_4)$. Hence, by Proposition 2.1 and Example 3.1, the projection map $\mathbb{R}^4 \ni x \mapsto x_1 + ix_2 \in S^2$ is critical and has energy $8\pi^2$.

To proceed further in our analysis of critical submersions, let us denote by ∇ both the Levi-Civita connexion on TM and on TN . Recall that the connexion on TN induces a connexion ∇^{ϕ} on $\phi^{-1}TN$. This connexion, together with the Levi-Civita connexion on TM , induces a connexion on $\text{Hom}(TM, \phi^{-1}TN)$, which we also denote by ∇ . The *second fundamental form* of ϕ is the covariant derivative of $d\phi$:

$$\nabla d\phi(X, Y) = \nabla_X^{\phi} d\phi(Y) - d\phi(\nabla_X Y) \quad (X, Y \in \Gamma(TM)).$$

The map ϕ is said to be *totally geodesic* if its second fundamental form vanishes. The *tension field* of ϕ is the trace of the second fundamental form:

$$\tau(\phi) = \text{trace } \nabla d\phi = \sum_{i=1}^m \nabla d\phi(e_i, e_i).$$

The map ϕ is said to be a *harmonic map* if its tension field vanishes.

To simplify our calculations, let us fix a point $x \in M$ and an orthonormal frame $\{e_i\}_{i=1}^m$ which is *normal* at x , i.e.,

$$\nabla_{e_i} e_j(x) = 0$$

for all i, j . Assuming that all calculations take place at the point $x \in M$, we can rewrite the Euler-Lagrange equations in the following way:

$$\begin{aligned}
d\phi(\sharp \delta \phi^* \omega) &= \sum_{j=1}^m d\phi(\delta \phi^* \omega(e_j) e_j) \\
&= -d\phi\left(\sum_{i,j=1}^m \nabla_{e_i}(\phi^* \omega(e_i, e_j) e_j)\right) \\
&= -\sum_{i,j=1}^m \left(e_i \omega(d\phi(e_i), d\phi(e_j))\right) d\phi(e_j) \\
&= \sum_{j=1}^m \left(\sum_{i=1}^m \omega(\nabla d\phi(e_i, e_j), d\phi(e_i)) - \omega(\tau(\phi), d\phi(e_j))\right) d\phi(e_j).
\end{aligned}$$

We thus define

$$(3.1) \quad S(\phi) = \sum_{j=1}^m \left(\sum_{i=1}^m \omega(\nabla d\phi(e_i, e_j), d\phi(e_i)) - \omega(\tau(\phi), d\phi(e_j))\right) d\phi(e_j);$$

by our calculation, the map ϕ is a critical point if and only if $S(\phi) = 0$.

Example 3.3. Assume that ϕ is a Riemannian submersion; thus, at each point $x \in M$, the differential

$$d\phi_x : T_x M \rightarrow T_{\phi(x)} N$$

maps the space $(\ker d\phi_x)^\perp \subseteq T_x M$ isometrically onto $T_{\phi(x)} N$. We first demonstrate that

$$\nabla d\phi(X, Y) = 0 \quad (X, Y \in (\ker d\phi_x)^\perp).$$

Any vector field X on N can be written as $d\phi(\hat{X})$ for some vector field \hat{X} on M taking values in $(\ker d\phi)^\perp$, and we can always find a local frame for $(\ker d\phi)^\perp$ of vector fields of the form \hat{X} . Thus, it is enough to show that

$$\nabla d\phi(\hat{X}, \hat{Y}) = 0 \quad (X, Y \in \Gamma(TN)).$$

Denoting by ∇^M and ∇^N the Levi-Civita connexions on M and N , respectively, we have for $X, Y, Z \in \Gamma(TN)$,

$$\begin{aligned}
g(\nabla_{\hat{X}}^M \hat{Y}, \hat{Z}) &= \frac{1}{2} \{ \hat{X} g(\hat{Y}, \hat{Z}) + \hat{Y} g(\hat{X}, \hat{Z}) - \hat{Z} g(\hat{X}, \hat{Y}) \\
&\quad + g([\hat{X}, \hat{Y}], \hat{Z}) + g([\hat{Z}, \hat{X}], \hat{Y}) - g([\hat{Y}, \hat{Z}], \hat{X}) \} \\
&= \frac{1}{2} \{ X h(Y, Z) + Y h(X, Z) - Z h(X, Y) \\
&\quad + h([X, Y], Z) + h([Z, X], Y) - h([Y, Z], X) \} \\
&= h(\nabla_X^N Y, Z) = g(\widehat{\nabla_X^N Y}, \hat{Z}).
\end{aligned}$$

Thus,

$$\nabla d\phi(\hat{X}, \hat{Y}) = \nabla_X^N Y - d\phi(\nabla_{\hat{X}}^M \hat{Y}) = 0 \quad (X, Y \in \Gamma(TN)).$$

Locally, we can choose an orthonormal frame e_1, \dots, e_m for TM with the property that e_1, \dots, e_{m-n} is a local frame for $\ker d\phi$ and e_{m-n+1}, \dots, e_m is a local frame for $(\ker d\phi)^\perp$. Then

$$S(\phi) = - \sum_{j=m-n+1}^m \omega(\tau(\phi), d\phi(e_j)) d\phi(e_j).$$

Thus, ϕ is critical if and only if ϕ is harmonic. In fact, using the same local frame for TM gives

$$\tau(\phi) = -d\phi\left(\sum_{j=1}^{m-n} \nabla_{e_j} e_j\right) = -d\phi(H),$$

where H is the mean curvature vector of the fibres of ϕ . We conclude that *a Riemannian submersion is a critical point if and only if it has minimal fibres, and thus is a harmonic morphism*, see [2]. Note that such a map, being harmonic, is automatically a critical point of the full Faddeev-Hopf functional for every value of the coupling, not just the infinite coupling limit.

For example, the natural projection

$$\phi : S^{2n+1} \rightarrow \mathbb{C}P^n, \quad \phi(z) = [z] \quad (z \in S^{2n+1} \subset \mathbb{C}^{n+1})$$

is a Riemannian submersion with minimal, even totally geodesic, fibres.

4. THE SECOND VARIATION AND STABILITY

In this section we calculate the second variation of the energy functional. Assume that $\phi : M \rightarrow N$ is a critical point of the functional. We define the *Hessian of E at ϕ* as

$$H_\phi(X, Y) = \frac{\partial^2}{\partial_s \partial_t} \Big|_{s=t=0} E(\phi_{s,t});$$

here $\phi_{s,t}$ is a 2-parameter variation of ϕ with

$$X = \partial_t \phi_{s,t} \Big|_{s=t=0} \quad \text{and} \quad Y = \partial_s \phi_{s,t} \Big|_{s=t=0}.$$

Clearly H_ϕ is a symmetric, bi-linear form on $\Gamma(\phi^{-1}TN)$. The map ϕ is said to be *stable* if

$$H_\phi(X, X) \geq 0 \quad (X \in \Gamma(\phi^{-1}TN));$$

the *index* of ϕ is the dimension of the largest subspace on which H_ϕ is negative.

Clearly, any map which minimizes the energy within its homotopy class is a stable critical point. In some situations it is easy to give lower bounds for the energy.

Proposition 4.1. *Let $M = M^2$ be a surface. Then*

$$E(\phi) \geq \frac{1}{2\text{Vol}(M)} \left(\int_M \phi^* \omega \right)^2.$$

Note that the right hand side is a homotopy invariant. If ϕ attains this lower bound then ϕ is either isotropic (so $E(\phi) = 0$) or has no critical points.

Proof. Since M is a surface, $\phi^* \omega = f * 1$ for some function f on M . By the Cauchy-Schwarz inequality we have

$$\int_M |f| * 1 \leq \|f\|_{L^2} \sqrt{\text{Vol}(M)},$$

with equality if and only if f is a constant. Hence

$$\begin{aligned} E(\phi) &= \frac{1}{2} \|f\|_{L^2}^2 \geq \frac{1}{2\text{Vol}(M)} \left(\int_M |f| * 1 \right)^2 \\ &\geq \frac{1}{2\text{Vol}(M)} \left(\int_M f * 1 \right)^2 = \frac{1}{2\text{Vol}(M)} \left(\int_M \phi^* \omega \right)^2, \end{aligned}$$

and the right hand side is a homotopy invariant. This lower bound is attained if and only if f is a constant. If $f = 0$ then $\phi^* \omega = 0$ so ϕ is isotropic. If $f \neq 0$ then ϕ has no critical points. \square

Example 4.2. Assume that $\phi : M^2 \rightarrow S^2$ is anisotropic and attains the bound. Then ϕ is necessarily a covering map; since S^2 is simply connected we must have $M = S^2$ and ϕ a diffeomorphism. But then $\int_M \phi^* \omega = \pm \text{Vol}(M)$, so $E(\phi) = \frac{1}{2} \text{Vol}(M)$. Thus $f \equiv \pm 1$, so ϕ is, up to an orientation reversing isometry, a symplectomorphism of S^2 . Hence the set of maps attaining the bound consists of the orbit of $\text{Id} : S^2 \rightarrow S^2$ under the group of symplectomorphisms of S^2 and the image of this orbit under reflexion.

Example 4.3. Let $M = N = T^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. Then the bound is attained in every homotopy class by

$$\phi(x, y) = (x, y)L \quad (L \in \text{Mat}_2(\mathbb{Z})).$$

Proposition 4.4. *Let $M = M^4$. Then*

$$E(\phi) \geq \frac{1}{2} \left| \int_M \phi^*(\omega \wedge \omega) \right|$$

with equality if and only if $\phi^ \omega$ is (anti-)self-dual. Note that the right hand side is a homotopy invariant.*

Proof. Since $\dim M = 4$, the Hodge $*$ -operator is an involution on the bundle of 2-forms. Thus any 2-form α can be decomposed as

$$\alpha = \alpha_+ + \alpha_-,$$

where $*\alpha_+ = \alpha_+$ and $*\alpha_- = -\alpha_-$. Since α_+ and α_- are mutually orthogonal we get

$$\|\alpha\|_{L^2}^2 = \|\alpha_+\|_{L^2}^2 + \|\alpha_-\|_{L^2}^2 \geq \left| \|\alpha_+\|_{L^2}^2 - \|\alpha_-\|_{L^2}^2 \right| = \left| \int_M \alpha \wedge \alpha \right|,$$

with equality if and only if $\alpha_+ = 0$ (α is anti-self-dual) or $\alpha_- = 0$ (α is self-dual). The proposition follows once we apply this to $\alpha = \phi^* \omega$. \square

Remark 4.5. The proposition tells us nothing useful if $H^2(M; \mathbb{R}) = 0$ or if $\dim N = 2$. In the first case $\phi^* \omega$ is necessarily exact, and hence so is $\phi^* \omega \wedge \phi^* \omega = \phi^*(\omega \wedge \omega)$, so we just recover the trivial fact that $E(\phi) \geq 0$. In the second case $\omega \wedge \omega = 0$, so again we deduce only $E(\phi) \geq 0$.

Example 4.6. In the case $M = N = T^4$, the bound is attained in each homotopy class, by linear maps with integer coefficients, as in Example 4.3.

Let us now find an explicit formula for the Hessian of a critical point. Note that the metric h on TN induces a metric, also denoted by h , on $\phi^{-1}TN$.

Proposition 4.7. *Assume that ϕ is a critical point of the energy functional. Then the Hessian of ϕ is given by*

$$H_\phi(X, Y) = \int_M h(X, \mathcal{L}_\phi Y) * 1 \quad (X, Y \in \Gamma(\phi^{-1}TN)),$$

where

$$\mathcal{L}_\phi Y = -J \left(\nabla_{Z_\phi}^\phi Y + d\phi(\sharp \delta d\phi^* \iota_Y \omega) \right) \quad \text{and} \quad Z_\phi = \sharp \delta \phi^* \omega.$$

Remark 4.8. Note that Z_ϕ is the vector field on M which must lie pointwise in $\ker d\phi$ given that ϕ is critical, by Corollary 2.4.

Proof of Proposition 4.7: Let $X_s = \partial_t \phi_{s,t}|_{t=0}$. Then, using the Homotopy Lemma and a calculation similar to that of the previous section,

$$(4.1) \quad \begin{aligned} \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} \frac{1}{2} \phi_{s,t}^* \omega \wedge * \phi_{s,t}^* \omega &= \frac{\partial}{\partial s} \Big|_{s=0} d\phi_s^* \iota_{X_s} \omega \wedge * \phi_s^* \omega \\ &= \left(\frac{\partial}{\partial s} \Big|_{s=0} d\phi_s^* \iota_{X_s} \omega \right) \wedge * \phi^* \omega + d\phi^* \iota_X \omega \wedge * d\phi^* \iota_Y \omega. \end{aligned}$$

When integrating over M , the second term on the right becomes

$$\langle d\phi^* \iota_X \omega, d\phi^* \iota_Y \omega \rangle_{L^2} = \langle \phi^* \iota_X \omega, \delta d\phi^* \iota_Y \omega \rangle_{L^2}.$$

Now note that, using a local orthonormal frame e_1, \dots, e_m for M , we get

$$\langle \phi^* \iota_X \omega, \delta d\phi^* \iota_Y \omega \rangle = \sum_{i=1}^m \phi^* \iota_X \omega(e_i) \delta d\phi^* \iota_Y \omega(e_i) = \omega(X, d\phi(\sharp \delta d\phi^* \iota_Y \omega)).$$

Let us now look at the first term on the right hand side of (4.1). Pointwise we have

$$\begin{aligned} \left\langle \frac{\partial}{\partial s} \Big|_{s=0} d\phi_s^* \iota_{X_s} \omega, \delta \phi^* \omega \right\rangle &= \sum_{i=1}^m \left(\frac{\partial}{\partial s} \Big|_{s=0} d\phi_s^* \iota_{X_s} \omega(e_i) \right) \delta \phi^* \omega(e_i) \\ &= \left(\frac{\partial}{\partial s} \Big|_{s=0} \phi_s^* \iota_{X_s} \omega \right) (Z_\phi) \\ &= \frac{\partial}{\partial s} \Big|_{s=0} \omega(X_s, d\phi_s(Z_\phi)) \\ &= \omega(\nabla_{\partial_s}^{\phi_s} X_s \Big|_{s=0}, d\phi(Z_\phi)) + \omega(X, \nabla_{\partial_s}^{\phi_s} d\phi_s(Z_\phi) \Big|_{s=0}). \end{aligned}$$

The first term on the right vanishes since we assume ϕ to be a critical point. The second term becomes

$$\omega(X, \nabla_{\partial_s}^{\phi_s} d\phi_s(Z_\phi) \Big|_{s=0}) = \omega(X, \nabla_{Z_\phi}^{\phi_s} d\phi_s(\partial_s) \Big|_{s=0}) = \omega(X, \nabla_{Z_\phi}^\phi Y),$$

where we used the fact that

$$\nabla_{\partial_s}^{\phi_s} d\phi_s(Z_\phi) = \nabla_{Z_\phi}^{\phi_s} d\phi_s(\partial_s) + d\phi_s([\partial_s, Z_\phi]) = \nabla_{Z_\phi}^{\phi_s} d\phi_s(\partial_s)$$

since $[\partial_s, Z_\phi] = 0$. □

Corollary 4.9. *Let $\phi : M \rightarrow N$ be a critical point of the energy functional. Then*

$$H_\phi(Y, Y) = \int_M \omega(Y, \nabla_{Z_\phi}^\phi Y) * 1 + \|d\phi^* \iota_Y \omega\|_{L^2}^2 \quad (Y \in \Gamma(\phi^{-1}TN)).$$

In particular, ϕ is stable if Z_ϕ vanishes.

Proof. Take a local orthonormal frame e_1, \dots, e_m for M . Then

$$\begin{aligned} h(JY, d\phi(\sharp \delta d\phi^* \iota_Y \omega)) &= \sum_{i=1}^m h(JY, d\phi(e_i)) \delta d\phi^* \iota_Y \omega(e_i) \\ &= \sum_{i=1}^m \phi^* \iota_Y \omega(e_i) \delta d\phi^* \iota_Y \omega(e_i) \\ &= \langle \phi^* \iota_Y \omega, \delta d\phi^* \iota_Y \omega \rangle. \end{aligned}$$

When integrated, this becomes $\|d\phi^* \iota_Y \omega\|_{L^2}^2$, and the proof follows. \square

Example 4.10. Assume that $\phi : M \rightarrow N$ is a critical immersion. According to Corollary 2.4, $Z_\phi = 0$, so ϕ is stable. In particular, the identity map of any compact Kähler manifold is stable. In the case $\dim N = 2$ or 4 , we have the stronger information that $\text{Id} : N \rightarrow N$ globally minimizes E within its homotopy class, by Propositions 4.1 and 4.4.

In fact, we can obtain some idea of $\text{spec } \mathcal{L}_{\text{Id}}$, the eigenvalue spectrum of \mathcal{L}_{Id} , from the following result. Note that

$$\mathcal{L}_{\text{Id}} = J^{-1} \flat^{-1} \delta d \flat J,$$

so that Y is an eigensection of \mathcal{L}_{Id} with eigenvalue λ if and only if $\flat JY$ is an eigenform of δd with eigenvalue λ .

Proposition 4.11. Denote by $\text{spec } \delta d$ and $\text{spec } d\delta$ the eigenvalue spectra of the operators

$$\delta d : \Gamma(T^*M) \rightarrow \Gamma(T^*M)$$

and

$$d\delta : \Gamma(\wedge^3 T^*M) \rightarrow \Gamma(\wedge^3 T^*M),$$

respectively. Then

$$\text{spec } \Delta_2 = \text{spec } \delta d \cup \text{spec } d\delta,$$

where Δ_2 denotes the Laplacian on 2-forms on M . Hence $\text{spec } \mathcal{L}_{\text{Id}} = \text{spec } \delta d \subseteq \text{spec } \Delta_2$ in general. Furthermore, if $\dim M = 2$ or $\dim M = 4$, then

$$(4.2) \quad \text{spec } \Delta_2 = \text{spec } \delta d = \text{spec } \mathcal{L}_{\text{Id}}.$$

Proof. Since the Kähler form on M is harmonic, $0 \in \text{spec } \Delta_2$. Assume that $\lambda \in \text{spec } \delta d$. To show that $\lambda \in \text{spec } \Delta_2$, we may thus assume that $\lambda \neq 0$. Then there is a 1-form $\alpha \neq 0$ on M with

$$\delta d\alpha = \lambda\alpha.$$

Then we must have $d\alpha \neq 0$ and so

$$d\delta d\alpha = \lambda d\alpha,$$

implying that

$$\Delta_2 d\alpha = \lambda d\alpha.$$

Hence $\text{spec } \delta d \subseteq \text{spec } \Delta_2$. The proof that $\text{spec } d\delta \subseteq \text{spec } \Delta_2$ is similar.

Conversely, assume $\lambda \in \text{spec } \Delta_2$. As $0 \in \text{spec } \delta d \cup \text{spec } d\delta$, we may assume that $\lambda \neq 0$. Then there is a 2-form $\xi \neq 0$ with

$$(4.3) \quad \Delta_2 \xi = \lambda \xi.$$

By the Hodge decomposition we can write

$$\xi = \xi_H + d\alpha + \delta\beta,$$

where ξ_H is a harmonic 2-form, α a 1-form and β a 3-form. By the Hodge decomposition of α and β we see that we may assume that α is coexact and β exact. From equation (4.3) it quickly follows that

$$\begin{aligned}\xi_H &= 0 \\ d\delta d\alpha &= \lambda d\alpha \\ \delta d\delta\beta &= \lambda \delta\beta.\end{aligned}$$

The second of these equations implies that the 1-form $\delta d\alpha - \lambda\alpha$ is closed; by assumption it is also coexact, and so it must vanish. If $\alpha \neq 0$ we thus have $\lambda \in \text{spec } \delta d$. On the other hand, the third equation implies that the 3-form $d\delta\beta - \lambda\beta$ is coclosed; by assumption it is also exact, so it must vanish. If $\beta \neq 0$ we thus have $\lambda \in \text{spec } d\delta$. Since at least one of α and β is non-zero, we must have

$$\lambda \in \text{spec } \delta d \cup \text{spec } d\delta.$$

The last statement is obvious when $\dim M = 2$ since any 3-form vanishes. When $\dim M = 4$, the action of $d\delta$ on 3-forms is equivalent to the action of δd on 1-forms under the Hodge isomorphism. \square

It is interesting to compare this with the behaviour of $\text{Id} : (M, g) \rightarrow (M, g)$ in harmonic map theory, where Id is not stable in general (though it is stable if M is Kähler) [8, 14]. The analogous operator to \mathcal{L}_{Id} is the Jacobi operator J_{Id} , whose spectral properties depend crucially on the Ricci curvature of M . There is a formula similar to (4.2),

$$\text{spec } J_{\text{Id}} = \text{spec } \Delta_1 - \frac{2s}{\dim M},$$

where s is the scalar curvature of M , but it holds only in the case that (M, g) is Einstein. More generally there is no simple relationship between $\text{spec } J_{\text{Id}}$ and $\text{spec } \Delta_p$. Analytically, J_{Id} is elliptic, and so has finite-dimensional kernel. By contrast, $\ker \mathcal{L}_{\text{Id}}$ is the space of symplectic vector fields (those Y for which $\iota_Y \omega$ is closed), which has infinite dimension. Clearly $\dim \ker \mathcal{L}_\phi = \infty$ for all critical maps due to the invariance of $E(\phi)$ under symplectic diffeomorphisms of N . In fact we will see in the next section an example of a critical map ϕ (the Hopf map $S^3 \rightarrow S^2$) for which *every* eigenspace of \mathcal{L}_ϕ has infinite dimension.

Example 4.12. In Example 3.1 we proved that the projection of a warped product

$$\phi : P \times_f N \rightarrow N$$

is critical when N is a Kähler manifold. This is not an immersion. However, we showed that $Z_\phi = 0$, so such a map is always stable nonetheless. The same is true of the critical projection $(\mathbb{R}^4, g) \rightarrow S^2$ of Example 3.2.

5. THE HOPF MAP

In this section we prove that the Hopf map $S^3 \rightarrow S^2$ is stable and calculate the spectrum of its Hessian. We then apply this to prove a conjecture of Ward regarding the full Faddeev-Hopf model.

We begin by introducing some Lie group and Lie algebra technicalities regarding symmetric and Hermitian symmetric spaces. We stringently follow the conventions used in [7], to which the reader is referred for definitions and fundamental results on symmetric spaces.

Assume that G is a compact, connected, simple Lie group and that K is a compact subgroup of G such that G/K is an irreducible Hermitian symmetric space of compact type. On the Lie algebra level we have the standard orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p},$$

where \mathfrak{k} is the Lie algebra of K and \mathfrak{p} an Ad_K -invariant subspace with the property that $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. It is well known that the Hermitian structure on G/K is induced by the adjoint action of an element in the centre of \mathfrak{k} ; in accordance with earlier notation, we denote this element by J .

We provide G with the Riemannian metric induced by the negative of the Killing form (or a suitable multiple thereof), and give G/K the metric which turns the homogeneous projection

$$\phi : G \rightarrow G/K, \quad g \mapsto g \cdot o$$

into a Riemannian submersion; here o denotes the identity coset in G/K . The fibres of ϕ are clearly minimal, even totally geodesic; according to Example 3.3, ϕ is a critical point of the functional. For simplicity, we denote by $\langle \cdot, \cdot \rangle$ the negative of the Killing form on \mathfrak{g} .

The pullback bundle $\phi^{-1}TG/K$ is isomorphic to the trivial bundle $G \times \mathfrak{p}$ by the map

$$G \times \mathfrak{p} \ni (g, X) \mapsto \left. \frac{d}{dt} \right|_{t=0} \phi(g) \exp tX \cdot o \in T_{\phi(g)}G/K;$$

the metric on $\phi^{-1}TG/K$ corresponds under this isomorphism to the metric on $G \times \mathfrak{p}$ induced by the restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{p} . Similarly, we can identify TG with the trivial bundle $G \times \mathfrak{g}$ by left translation, and this gives the following commutative diagram:

$$\begin{array}{ccc} G \times \mathfrak{g} & \xrightarrow{\cong} & TG \\ \downarrow & & \downarrow d\phi \\ G \times \mathfrak{p} & \xrightarrow{\cong} & \phi^{-1}TG/K \end{array}$$

The map on the left, which we thus identify with $d\phi$, is induced by orthogonal projection

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \rightarrow \mathfrak{p}.$$

With this identification in mind, we think of sections of TG as functions on G with values in \mathfrak{g} and sections of $\phi^{-1}TG/K$ as functions on G with values in \mathfrak{p} :

$$\Gamma(TG) \cong C^\infty(G, \mathfrak{g}), \quad \Gamma(\phi^{-1}TG/K) \cong C^\infty(G, \mathfrak{p}).$$

The sections of TG are of course also derivations: for any vector space V and smooth function $f : G \rightarrow V$, an element $X \in C^\infty(G, \mathfrak{g})$ acts on f as

$$X(f)(g) = df(X)(g) = \left. \frac{d}{dt} \right|_{t=0} f(g \exp(tX(g))) \quad (g \in G).$$

The Levi-Civita connexion on TG corresponds to the connexion

$$\nabla_X Y = dY(X) + \frac{1}{2}[X, Y] \quad (X, Y \in C^\infty(G, \mathfrak{g})),$$

and the pullback of the Levi-Civita connexion on TG/K to $\phi^{-1}TG/K$ corresponds to the connexion

$$\nabla_X^\phi Y = dY(X) + [X, Y] \quad (X \in C^\infty(G, \mathfrak{g}), Y \in C^\infty(G, \mathfrak{p})).$$

Proposition 5.1. *Choose an orthonormal basis $\{e_k\}_{k=1}^m$ for \mathfrak{g} such that e_1, \dots, e_{m-n} is a basis for \mathfrak{k} and e_{m-n+1}, \dots, e_m a basis for \mathfrak{p} . For the homogeneous projection ϕ , the second variation takes the form*

$$\begin{aligned} \mathcal{L}_\phi Y = -J \Bigg(& -\frac{\lambda}{2} J(Y) - J \sum_{k=1}^m e_k e_k(Y) + \frac{3}{2} J \sum_{a=1}^n [e_a(Y), e_a] \\ & + \sum_{r,s=m-n+1}^m \omega(e_r e_s(Y) - \frac{1}{2}[e_r, e_s](Y), e_r) e_s \Bigg). \end{aligned}$$

Here λ is the eigenvalue of the Casimir operator associated to the adjoint representation of \mathfrak{g} :

$$-\sum_{k=1}^m [e_k, [e_k, X]] = \lambda X \quad (X \in \mathfrak{g}).$$

Proof. We begin by calculating Z_ϕ . By Corollary 2.4 we know that $\delta\phi^*\omega(X) = 0$ for $X \in \mathfrak{p}$, and for $X \in \mathfrak{k}$ we have

$$\begin{aligned} \delta\phi^*\omega(X) &= \sum_{k=1}^m \left(-e_k(\phi^*\omega(e_k, X)) + \phi^*\omega(\nabla_{e_k} e_k, X) + \phi^*\omega(e_k, \nabla_{e_k} X) \right) \\ &= \sum_{k=1}^m \omega(d\phi(e_k), d\phi(\nabla_{e_k} X)) \\ &= \sum_{r=m-n+1}^m \omega(d\phi(e_r), d\phi(\frac{1}{2}[e_r, X])) \\ &= \frac{1}{2} \sum_{r=m-n+1}^m \omega(e_r, [e_r, X]) \\ &= \frac{1}{2} \sum_{r=m-n+1}^m \langle [Je_r, e_r], X \rangle. \end{aligned}$$

Thus,

$$Z_\phi = \frac{1}{2} \sum_{r=m-n+1}^m [Je_r, e_r] = \frac{1}{2} \sum_{r=m-n+1}^m [e_r, [e_r, J]] = \frac{1}{2} \sum_{k=1}^m [e_k, [e_k, J]] = -\frac{\lambda}{2} J,$$

where we have used the fact that J belongs to the centre of \mathfrak{k} .

Next we look at $\phi^* \iota_Y \omega$. Let $A, B \in \mathfrak{g}$. Then

$$\begin{aligned} d\phi^* \iota_Y \omega(A, B) &= A(\omega(Y, d\phi(B))) - B(\omega(Y, d\phi(A))) - \omega(Y, d\phi([A, B])) \\ &= \omega(dY(A), B_{\mathfrak{p}}) - \omega(dY(B), A_{\mathfrak{p}}) - \omega(Y, [A, B]_{\mathfrak{p}}). \end{aligned}$$

Thus, for $X \in \mathfrak{p}$,

$$\begin{aligned} \delta d\phi^* \iota_Y \omega(X) &= \sum_{k=1}^m \left(-e_k(d\phi^* \iota_Y \omega(e_k, X)) + d\phi^* \iota_Y \omega(\nabla_{e_k} e_k, X) \right. \\ &\quad \left. + d\phi^* \iota \omega(e_k, \nabla_{e_k} X) \right) \\ &= \sum_{k=1}^m \left(-e_k(\omega(dY(e_k), X) - \omega(dY(X), (e_k)_{\mathfrak{p}}) - \omega(Y, [e_k, X]_{\mathfrak{p}})) \right. \\ &\quad \left. + \omega(dY(e_k), (\nabla_{e_k} X)_{\mathfrak{p}}) - \omega(dY(\nabla_{e_k} X), (e_k)_{\mathfrak{p}}) - \omega(Y, [e_k, \nabla_{e_k} X]_{\mathfrak{p}}) \right) \\ &= \sum_{k=1}^m \left(-\omega(e_k e_k(Y), X) - \frac{1}{2} \omega(Y, [e_k, [e_k, X]]_{\mathfrak{p}}) \right) \\ &\quad + \frac{3}{2} \sum_{a=1}^{m-n} \omega(e_a(Y), [e_a, X]) \\ &\quad + \sum_{r=m-n+1}^m \left(\omega(e_r X(Y), e_r) - \frac{1}{2} \omega([e_r, X](Y), e_r) \right) \\ &= \left\langle -J \sum_{k=1}^m e_k e_k(Y) + \frac{\lambda}{2} JY + \frac{3}{2} \sum_{a=1}^{m-n} J[e_a(Y), e_a], X \right\rangle \\ &\quad + \sum_{r=m-n+1}^m \omega((e_r X - \frac{1}{2}[e_r, X])(Y), e_r). \end{aligned}$$

Hence

$$\begin{aligned} d\phi(\# \delta d\phi^* \iota_Y \omega) &= -J \sum_{k=1}^m e_k e_k(Y) + \frac{\lambda}{2} JY + \frac{3}{2} \sum_{a=1}^{m-n} J[e_a(Y), e_a] \\ &\quad + \sum_{r,s=m-n+1}^m \omega((e_r e_s - \frac{1}{2}[e_r, e_s])(Y), e_r) e_s. \end{aligned}$$

We thus arrive at

$$\begin{aligned} \mathcal{L}_\phi(Y) &= -J \left(-\frac{\lambda}{2} J(Y) - J \sum_{k=1}^m e_k e_k(Y) + \frac{3}{2} \sum_{a=1}^{m-n} J[e_a(Y), e_a] \right. \\ &\quad \left. + \sum_{r,s=m-n+1}^m \omega((e_r e_s - \frac{1}{2}[e_r, e_s])(Y), e_r) e_s \right). \end{aligned}$$

□

The Hopf map is by definition the map

$$\phi : S^3 \subset \mathbb{C}^2 \rightarrow \mathbb{C}P^1, \quad \phi(z_1, z_2) = [z_1, z_2].$$

By the identifications

$$S^3 \cong \mathbf{SU}(2), \quad \mathbb{CP}^1 \cong \mathbf{SU}(2)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(1)),$$

we get the alternative definition of the Hopf map as the homogeneous projection

$$\phi : \mathbf{SU}(2) \rightarrow \mathbf{SU}(2)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(1)).$$

Using the previous result and some representation theory, we shall prove the following result:

Theorem 5.2. *The Hopf map is stable; the Hessian has eigenvalues $\frac{1}{4}(n^2 + 2n)$ and $\frac{1}{4}(n - 2k)^2$, $k = 0, \dots, n$, $n = 1, 2, \dots$. Each eigenspace is of infinite dimension.*

It is well known that, as a harmonic map, the Hopf map is unstable, see [16]. Returning to the full Faddeev-Hopf model for maps $\mathbf{SU}(2) \rightarrow \mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(1))$,

$$E(\phi) = \frac{1}{2} \int_{\mathbf{SU}(2)} (|\mathrm{d}\phi|^2 + \alpha |\phi^* \omega|^2) * 1,$$

we can give precise information on the stability of the Hopf map for this functional, thus proving a conjecture stated by Ward in [17]. The stability properties turn out to be exactly analogous to those of the identity map in the Skyrme model on S^3 [10].

Theorem 5.3. *The Hopf map is an unstable critical point of the Faddeev-Hopf energy functional if $\alpha < 1$ and a stable critical point if $\alpha \geq 1$.*

The remaining part of this section is devoted to the proof of these two results, beginning with Theorem 5.2.

The Lie algebra $\mathfrak{su}(2)$ of $\mathbf{SU}(2)$ has a basis

$$\vartheta_1 = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vartheta_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \vartheta_3 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and we choose the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{su}(2)$ which makes this an orthonormal basis; then $\langle \cdot, \cdot \rangle$ is a multiple of the Killing form. The isotropy subalgebra $\mathfrak{k} = \mathfrak{s}(\mathbf{u}(1) \times \mathbf{u}(1))$ is one-dimensional and spanned by ϑ_3 , which also acts as J on $\mathfrak{p} = \text{span}\{\vartheta_1, \vartheta_2\}$.

It is easy to see that, for the adjoint representation of $\mathfrak{su}(2)$, the Casimir operator is just multiplication by 2, i.e., $\lambda = 2$. To calculate \mathcal{L}_ϕ , let $f \in C^\infty(G, \mathbb{R})$. Then

$$\begin{aligned}
\mathcal{L}_\phi(f\vartheta_1) &= -J \left(-\vartheta_3(f)\vartheta_1 - (\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2)(f)[\vartheta_3, \vartheta_1] + \frac{3}{2}\vartheta_3(f)[\vartheta_3, [\vartheta_1, \vartheta_3]] \right. \\
&\quad \left. + (\vartheta_2\vartheta_1 - \frac{1}{2}[\vartheta_2, \vartheta_1])(f)\langle [\vartheta_3, \vartheta_1], \vartheta_2 \rangle \vartheta_1 + \vartheta_2^2(f)\langle [\vartheta_3, \vartheta_1], \vartheta_2 \rangle \vartheta_2 \right) \\
&= -J \left(-\vartheta_3(f)\vartheta_1 + (\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2)(f)\vartheta_2 + \frac{3}{2}\vartheta_3(f)\vartheta_1 \right. \\
&\quad \left. - \vartheta_2\vartheta_1(f)\vartheta_1 + \frac{1}{2}\vartheta_3(f)\vartheta_1 - \vartheta_2^2(f)\vartheta_2 \right) \\
&= -J \left((\vartheta_3(f) - \vartheta_2\vartheta_1(f))\vartheta_1 + (\vartheta_1^2 + \vartheta_3^2)(f)\vartheta_2 \right) \\
&= (-\vartheta_1^2 - \vartheta_3^2)(f)\vartheta_1 + (\vartheta_3 - \vartheta_2\vartheta_1)(f)\vartheta_2.
\end{aligned}$$

A similar calculation gives

$$\mathcal{L}_\phi(f\vartheta_2) = (-\vartheta_3 - \vartheta_1\vartheta_2)(f)\vartheta_1 + (-\vartheta_2^2 - \vartheta_3^2)(f)\vartheta_2.$$

Hence we can express the differential operator \mathcal{L}_ϕ as a matrix using the basis $\{\vartheta_1, \vartheta_2\}$ for \mathfrak{p} :

$$\mathcal{L}_\phi = \begin{pmatrix} -\vartheta_1^2 - \vartheta_3^2 & -\vartheta_3 - \vartheta_1\vartheta_2 \\ \vartheta_3 - \vartheta_2\vartheta_1 & -\vartheta_2^2 - \vartheta_3^2 \end{pmatrix}.$$

To calculate the spectrum of \mathcal{L}_ϕ we recall the Peter-Weyl theorem [9, p17]. According to this, $L^2(\mathbf{SU}(2), \mathbb{R})$ is the orthogonal sum of the finite-dimensional subspaces spanned by matrix elements for the (finite-dimensional) irreducible unitary representations of $\mathbf{SU}(2)$. Furthermore, these subspaces are invariant under \mathcal{L}_ϕ . To calculate the spectrum of \mathcal{L}_ϕ it is therefore enough to calculate the spectrum of \mathcal{L}_ϕ when restricted to these subspaces. Let us therefore momentarily digress for a study of the irreducible representations of $\mathbf{SU}(2)$.

As $\mathbf{SU}(2)$ is the compact real form of $\mathbf{SL}_2(\mathbb{C})$, all irreducible representations of $\mathbf{SU}(2)$ are obtained by restriction of the irreducible representations of $\mathbf{SL}_2(\mathbb{C})$. For a basis of $\mathfrak{sl}_2(\mathbb{C})$, let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Denote by $V = \mathbb{C}^2$ the standard representation of $\mathbf{SL}_2(\mathbb{C})$ and by $V^{(n)} = \text{Sym}^n(V)$ the n^{th} symmetric power of V , $n = 1, 2, \dots$. These are precisely the irreducible, finite-dimensional representations of $\mathbf{SL}_2(\mathbb{C})$, and therefore also of $\mathbf{SU}(2)$. To study the action of $\mathfrak{su}(2)$ on $V^{(n)}$, recall that there is a highest weight vector $v \in V^{(n)}$; let

$$v_k = Y^k v \quad (k = 0, 1, \dots, n).$$

We adopt the convention that $v_k = 0$ for $k < 0$ and $k > n$. Then, if v is suitably chosen,

$$Hv_k = (n - 2k)v_k, \quad Xv_k = k(n - k + 1)v_{k-1}, \quad Yv_k = v_{k+1}.$$

Furthermore, $\{v_k\}_{k=0}^n$ is a basis of $V^{(n)}$. A simple calculation gives that

$$\begin{aligned}\vartheta_1 v_k &= \frac{i}{2}(k(n-k+1)v_{k-1} + v_{k+1}) \\ \vartheta_2 v_k &= \frac{1}{2}(k(n-k+1)v_{k-1} - v_{k+1}) \\ \vartheta_3 v_k &= \frac{i}{2}(n-2k)v_k \\ (-\vartheta_1^2 - \vartheta_2^2)v_k &= \frac{1}{2}(2kn - 2k^2 + n)v_k \\ (-\vartheta_1^2 - \vartheta_2^2 - \vartheta_3^2)v_k &= \frac{1}{4}(n^2 + 2n)v_k.\end{aligned}$$

Let us now return to \mathcal{L}_ϕ and study its action on $V^{(n)} \otimes \mathfrak{p}$, where we think of ϑ_k as acting by the representation. Since $\vartheta_3 = [\vartheta_2, \vartheta_1]$, we can rewrite \mathcal{L}_ϕ as

$$(5.1) \quad \mathcal{L}_\phi = (-\vartheta_1^2 - \vartheta_2^2 - \vartheta_3^2)Id + \begin{pmatrix} \vartheta_2^2 & -\vartheta_2\vartheta_1 \\ -\vartheta_1\vartheta_2 & \vartheta_1^2 \end{pmatrix}.$$

Let us denote by $A^{(n)}$ the second operator, as acting on $V^{(n)} \otimes \mathfrak{p}$. Then, by our earlier calculations,

$$(5.2) \quad \mathcal{L}_\phi = \frac{1}{4}(n^2 + 2n)Id + A^{(n)}.$$

To find the eigenvalues of \mathcal{L}_ϕ on $V^{(n)} \otimes \mathfrak{p}$, we must find the eigenvalues of $A^{(n)}$. So assume that $\alpha \otimes \vartheta_1 + \beta \otimes \vartheta_2 \cong (\alpha, \beta)$ is an eigenvector with eigenvalue λ . Then

$$\begin{cases} \vartheta_2^2 \alpha - \vartheta_2 \vartheta_1 \beta &= \lambda \alpha \\ -\vartheta_1 \vartheta_2 \alpha + \vartheta_1^2 \beta &= \lambda \beta \end{cases} \implies (\vartheta_1^2 + \vartheta_2^2)(\vartheta_2 \alpha - \vartheta_1 \beta) = \lambda(\vartheta_2 \alpha - \vartheta_1 \beta).$$

Again, by our earlier calculations, we see that the only possibility is that either $\lambda = 0$, or

$$\lambda = \lambda_k = -\frac{1}{2}(2kn - 2k^2 + n) \quad (k = 0, \dots, n).$$

Furthermore, when $\alpha = \frac{1}{\lambda_k} \vartheta_2 v_k$ and $\beta = -\frac{1}{\lambda_k} \vartheta_1 v_k$, then it is easy to see that $\alpha \otimes \vartheta_1 + \beta \otimes \vartheta_2$ is an eigenvector for $A^{(n)}$ with eigenvalue λ_k . As the linear map

$$V^{(n)} \otimes \mathfrak{p} \rightarrow V^{(n)}, \quad (\alpha, \beta) \mapsto \vartheta_2 \alpha - \vartheta_1 \beta$$

is surjective, the dimension of its kernel equals $\dim V^{(n)} = n+1$. Hence we conclude that \mathcal{L}_ϕ , acting on $V^{(n)} \otimes \mathfrak{p}$, has the eigenvalue $\frac{1}{4}(n^2 + 2n)$ of multiplicity $n+1$, and the eigenvalues

$$\frac{1}{4}(n^2 + 2n) - \frac{1}{2}(2kn - 2k^2 + n) = \frac{1}{4}(n - 2k)^2 \quad (k = 0, \dots, n),$$

each of multiplicity 1.

To calculate the spectrum of \mathcal{L}_ϕ , this time acting as a differential operator on the space spanned by the matrix elements for the representation $V^{(n)} \otimes \mathfrak{p}$ of $\mathbf{SU}(2)$, choose some $\mathbf{SU}(2)$ -invariant inner product (\cdot, \cdot) on $V^{(n)}$ and define the functions

$$\pi_{kl}(g) = (gv_k, v_l) \quad (g \in \mathbf{SU}(2), \quad k, l = 0, \dots, n).$$

Then, by the invariance of (\cdot, \cdot) ,

$$\vartheta(\pi_{kl})(g) = (g\vartheta v_k, v_l) \quad (g \in \mathbf{SU}(2), \vartheta \in \mathfrak{su}(2), k, l = 0, \dots, n).$$

It follows that, for $k = 0, \dots, n$,

$$\frac{1}{\lambda_k} \vartheta_2(\pi_{kl}) \otimes \vartheta_1 - \frac{1}{\lambda_k} \otimes \vartheta_1(\pi_{kl}) \otimes \vartheta_2 \quad (l = 0, \dots, n)$$

is an eigenfunction of \mathcal{L}_ϕ with eigenvalue $\frac{1}{4}(n-2k)^2$; the corresponding eigenspace is of dimension $n+1$. Furthermore, it is clear that the kernel of $A^{(n)}$, acting on $\text{span}\{\pi_{kl}\}_{k,l=0}^n \otimes \mathfrak{p}$, is of dimension $(n+1)^2$; the action of \mathcal{L}_ϕ on $\ker A^{(n)}$ is therefore diagonal with eigenvalue $\frac{1}{4}(n^2 + 2n)$.

Thus \mathcal{L}_ϕ has the eigenvalue spectrum claimed. Further, we have shown that \mathcal{L}_ϕ is non-negative on an L^2 orthogonal collection of finite-dimensional spaces spanning L^2 . Hence $\langle \cdot, \mathcal{L}_\phi \cdot \rangle$ is non-negative on L^2 , so ϕ is stable. This completes the proof of Theorem 5.2.

Proof of Theorem 5.3. Following Urakawa [16], we denote by D the Jacobi operator of the Hopf map ϕ with respect to the Dirichlet energy. The Hessian of ϕ with respect to the full Faddeev-Hopf functional

$$E(\phi) = \frac{1}{2} \int_{\mathbf{SU}(2)} (|\mathrm{d}\phi|^2 + \alpha |\phi^* \omega|^2) * 1$$

is then obviously

$$\int_{\mathbf{SU}(2)} h((D + \alpha \mathcal{L}_\phi)X, Y) * 1 \quad (X, Y \in C^\infty(\mathbf{SU}(2), \mathfrak{p})),$$

where, as before,

$$\mathfrak{p} = \text{span}\{\vartheta_1, \vartheta_2\} \subset \mathfrak{su}(2).$$

We begin by studying the action of $D + \alpha \mathcal{L}_\phi$ on the spaces $V^{(n)} \otimes \mathfrak{p}$. From [16] and Theorem 5.2 it follows that $D + \alpha \mathcal{L}_\phi$ is positive semi-definite for $n \neq 1$ and all $\alpha \geq 0$. According to [16, Corollary 8.12] and (5.1) and (5.2) we have on $V^{(1)} \otimes \mathfrak{p}$

$$\begin{aligned} D + \alpha \mathcal{L}_\phi &= (1 + \alpha)(-\vartheta_1^2 - \vartheta_2^2 - \vartheta_3^2)Id - 2 \begin{pmatrix} 0 & \vartheta_3 \\ -\vartheta_3 & 0 \end{pmatrix} + \alpha A^{(1)} \\ &= \frac{3}{4}(1 + \alpha)Id + \begin{pmatrix} \alpha \vartheta_2^2 & -2\vartheta_3 - \alpha \vartheta_2 \vartheta_1 \\ 2\vartheta_3 - \alpha \vartheta_1 \vartheta_2 & \alpha \vartheta_1^2 \end{pmatrix}. \end{aligned}$$

Let $\{e_1, e_2\}$ be the standard basis for $V^{(1)} \cong \mathbb{C}^2$. In the basis $\{e_1 \otimes \vartheta_1, e_1 \otimes \vartheta_2, e_2 \otimes \vartheta_1, e_2 \otimes \vartheta_2\}$ for $V^{(1)} \otimes \mathfrak{p}$ we can express the last term as the matrix

$$\begin{pmatrix} -\alpha/4 & -i - i\alpha/4 & 0 & 0 \\ i + i\alpha/4 & -\alpha/4 & 0 & 0 \\ 0 & 0 & -\alpha/4 & -i - i\alpha/4 \\ 0 & 0 & i + i\alpha/4 & -\alpha/4 \end{pmatrix}.$$

This matrix has eigenvalues 1 and $-\frac{\alpha}{2} - 1$, each with multiplicity 2. On the eigenspace corresponding to the eigenvalue 1, the operator $D + \alpha \mathcal{L}_\phi$ is obviously

positive semi-definite for all α , while on the eigenspace corresponding to the eigenvalue $-\frac{\alpha}{2} - 1$, we have

$$D + \alpha \mathcal{L}_\phi = \frac{3}{4}(1 + \alpha)Id - \left(\frac{\alpha}{2} + 1\right)Id = \frac{\alpha - 1}{4}Id.$$

We thus conclude that for $\alpha \geq 1$, $D + \alpha \mathcal{L}_\phi$ is positive semi-definite on $V^{(n)} \otimes \mathfrak{p}$ for all n , while for $\alpha < 1$, it is negative definite on a 2-dimensional subspace of $V^{(1)} \otimes \mathfrak{p}$. Hence, the operator $D + \alpha \mathcal{L}_\phi$, acting on the space spanned by the matrix elements for the representation $V^{(n)} \otimes \mathfrak{p}$ of $\mathbf{SU}(2)$, is positive semi-definite for all n if $\alpha \geq 1$, while it is negative definite on a 4-dimensional subspace if $\alpha < 1$. Theorem 5.3 now follows from the Peter-Weyl Theorem. \square

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